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APPROXIMATION BY A SUM OF COMPLEX EXPONENTIALS

UTILIZING THE PENCIL OF FUNCTION METHOD

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ABSTRACT: The approximation of a function by a sum of complex exponentials, in general, is a nonlinear optimization problem. The optimization problem, however, is linearized through the application of the pencil of function method. This non-iterative method yields the best exponential approximation for a given order of approximation. The method differs radically from the classical Wiener least squares approach in the sense that exponents calculated by the pencil of function method are directly proportional to the integrated squared error in the approximation. As the integrated squared error approaches zero, the exponents calculated by the pencil of function method approach the best least squares exponents in a continuous fashion. Among the advantages of the method are its natural insensitivity to noise in the data and explicit determination of the signal order. Examples are presented to illustrate the stability of this technique especially when noise is present in the data.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The approximation of a function by a sum of complex exponentials, in general, is a nonlinear optimization problem. The optimization problem, however, is linearized through the application of the pencil of function method. This non-iterative method yields the best exponential approximation for a given order of approximation. The method differs radically from the classical Wiener least squares approach in the sense that exponents calculated by the pencil of function method are directly proportional to the integrated squared error in the approximation. As the integrated squared error approaches zero,

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best least squares exponents in a continuous fashion. Among the advantages
of the method are its natural insensitivity to noise in the data and
explicit determination of the signal order. Examples are presented to
illustrate the stability of this technique especially when noise is present
in the data.

## I. INTRODUCTION

The approximation of an arbitrary real function $x(t)$ which is nonzero for $t \geq 0$, by a sum of complex exponentials often leads to a nonlinear least squares problem. Recently Goulb [1] has presented a class of least squares problems in which the variables separate. One of the objectives of this paper is to show that the pencil of function method also decouples the approximation problem [2-3]. In this presentation we apply the pencil of function method to both the continuous function $x(t)$ and $y(t)$, which is the discretized version of $x(t)$. We show that the resulting expressions are quite different for the two cases. First we present the discrete approximation problem followed by the continuous case. Finally examples are presented to illustrate the stability of this method in the calculation of the exponentials particularly for noisy data.

## 2. Definition of the Problem

The problem of interest is to approximate an arbitrary square integrable function $x(t)$ by a sum of complex exponentials, i.e.

$$x(t) \approx \sum_{i=1}^{\ell} \sum_{k=1}^{m_i} A_{ik} \cdot \{t\}^{k-1} \cdot \exp\{s_i t\} \qquad (1)$$

where

$$\sum_{i=1}^{\ell} m_i = n \qquad (2)$$

Here $s_i$ are referred to as the poles of the system and $m_i$ is the order of the pole $s_i$. $n$ is the total number of poles counting the multiplicities of repeated poles.

In most problems, however, we do not deal with continuous functions but with sampled sequences. So we will assume that the sequence $\{y(p)\}$ has been sampled at a sufficiently fast rate from the continuous function $x(t)$ so that no information has been lost. Thus for a sampled data system we have

$$y(p) = \sum_{i=1}^{\ell} \sum_{k=1}^{m_i} A_{ik} \{p\}^{k-1} \{z_i\}^p \tag{3}$$

where

$$z_i = \exp\{s_i T\} \tag{4}$$

where $T$ is the sampling interval. Here $z_i$ are referred to as the z-domain poles.

The problem of approximating $A_{ik}$ and $s_i$ in (1) or $A_{ik}$ and $z_i$ in (3) is a nonlinear approximation problem. However we can linearize this nonlinear problem through the introduction of the T operator.

3. Properties of the T Operator.

The T operator is defined to be an operator similar to a reverse integral operator. It is defined on the space of squared integrable functions in the following way:

$$T[Y] \triangleq T\{y(p)\} \triangleq \{ \sum_{k=p}^{\infty} y(k) \} \tag{5}$$

so for example

$$T\{0,1,0,0,\ldots\} = \{1,1,0,0,0,\ldots\} \tag{6}$$

and

$$T\{p(0.5)^p\} = \{2p(0.5)^p + 2(0.5)^p\} \tag{7}$$

From the definition of T in (5) we can prove the following:

(a) T is a linear operator, since

$$T[\alpha Y + \beta Z] = \alpha T[Y] + \beta T[Z] \tag{8}$$

(2)

where $\alpha$ and $\beta$ are scalar quantities.  [Y] and [Z] are two sampled data sequences  as defined in (5) .

(b) A constant multiplication of a unit impulse remains unchanged by the application of the T operator, i.e.

$$T[a\delta] \triangleq T\{a,0,0,\ldots\} = aT\{1,0,0,\ldots\} = aT[\delta] \tag{9}$$

(c) for simple poles (i.e. $m_i=1$ for all i) we have

$$T[Y] = T\{y(p)\} = T\{\sum_{i=1}^{n} A_i(z_i)^P\} = \{\sum_{i=1}^{n} A_i \frac{(z_i)^P}{1-z_i} \} \tag{10}$$

In particular we have

$$T\{A(z) \} = \{\frac{A(z)^P}{1-z}\} \tag{11}$$

(d)  Successive operations of T are obtained in the following way:

$$T^0 = 1$$

$$T^k[Y] = T^{k-1}[TY] \tag{12}$$

and thus we can generate a polynomial in T.  One such polynomial, which will be of fundamental importance, is

$$[1-(1-z)T \ ]^k$$

(e)  Also, it can be shown that

$$(1-z)T\{(p)^i(z)^P\} = (p)^i(z)^P+z\sum_{k=0}^{i=1} (_k^i)T\{(p)^k(z)^P\} \tag{13}$$

where

$$(_k^i) = \frac{i!}{k!(i-k)!} \tag{14}$$

Properties (c) and (e) demonstrate that the application of the operator T on the sequence $\{y(p)\}$ preserves the poles of the sequence.

## 4. Properties of the $[1-(1-z)T]^k$ Polynomials:

From property (c) it is clear that if $\{y(p)\}$ be a sequence having a pole z repeated with multiplicity m, then the multiplicity of z in the sequences $[1-(1-z)T]^k\{y(p)\}$ is m-k, k=0,1,2,...,m-k     (15)

Furthermore,

$$[1-(1-z)T]^k \{y(p)\} = \{0\} \qquad \text{for } k \geq m \qquad (16)$$

Geometrically, the above property asserts that the operator $[1-(1-z)T]^k$ projects Y into a lower dimensional hyperplane in z of order m-k. Hence, it follows that, if the operator

$$\prod_{i=1}^{n} [1-(1-z)T]^{m_i} \qquad (17)$$

is applied to [Y] which is given by

$$[Y] = \{y(p)\} = \{ \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_{ik}(p)^{k-1}(z_i)^p \} \qquad (18)$$

then the result would be the null sequence $\{0\}$. We have thus linearized the problem by first attempting to estimate the poles $z_i$. After the poles $z_i$ have been computed, the residues $A_{ij}$ at the poles can be obtained by a least squares procedure.

5. <u>Determination of the Poles $z_i$.</u>

In order to determine the poles $z_i$, we define the following operations. Let

$$[Y_2] \triangleq T[Y_1] \triangleq T[TY_0] = T^2[Y_0] \qquad (19)$$

and

$$[Z_i] \triangleq [1-(1-z)T][Y_i] = [1+dT][Y_i] \qquad (20)$$

where

$$d \triangleq -(1-z) \qquad (21)$$

From (20) it is seen that

$$[Z_i] = [Y_i]+d[T][Y_i]=[T]\{Y_{i-1}+d[T][Y_{i-1}]\} = [T][Z_{i-1}] \quad (22)$$

The sequence $[Z_i]$ is called a pencil of functions parameterized by the scalar parameter d [2-5]. It is a linear combination of functions $[Y_i]$ and $[Y_{i+1}]$ through the parameter d. The pencil of functions contain very important characteristics from a system identification

(4)

point of view. When the set $[Z_i]$ of n+1 functions becomes linearly dependent then (d+1) becomes a system pole. Now we define the gram matrix $[\underline{H}]$ whose ith row and jth column are defined as

$$[\underline{H}] = \left( h_{ij} \triangleq <Z_i, Z_j> = <[1+dT][Y_i], [1+dT][Y_j]> \right)$$

$$= [d*\{d<Y_{i+1}, Y_{j+1}> + <Y_i, Y_{j+1}> + d<Y_{i+1}, Y_j> + <Y_i, Y_j>\}] \qquad (23)$$

where

$$<Y_i, Y_j> = \sum_{p=0}^{\infty} \{y_i(p)\}\{y_j(p)\} \qquad (24)$$

and d* denotes the complex conjugate of d. The underbar distinguishes a matrix from a sequence.

The significance of the gram matrix $[\underline{H}]$ lies in the fact that the set $[Z_1]$, $[Z_2]$, $[Z_3]$ ... $[Z_k]$ are linearly independent if and only if $\det[\underline{H}]>0$, whereas it is linearly dependent if and only if $\det[\underline{H}]=0$. Thus in order that there are n poles in the sequence $[Z_i]$ it is necessary and sufficient that $\det[\underline{H}]$ be positive or zero according as $k<n$, or $k\geq n$. [This is clear from (16)]

After some very tedious algebraic manipulations it can be shown that

$$\det[\underline{H}] = \sum_{i=1}^{k} \sum_{j=1}^{k} (d*)^{i-1} (d)^{j-1} Q_{ij} \qquad (25)$$

where $Q_{ij}$ are the diagonal cofactors of the matrix $[\underline{G}]$ whose elements are defined as

$$[\underline{G}] = [g_{ij} \triangleq <Y_i, Y_j> = \sum_{p=0}^{\infty} \{y_i(p)\}\{y_j(p)\}] \qquad (26)$$

An on line procedure for determining (26) is given by the following equations:

$$x_i(k) = y_i(k)$$

$$x_i(k) = \sum_{\nu=k}^{m-1} x_{i-1}(\nu)$$

(5)

$$g_{ij} = (-1)^{i+j} \sum_{k=0}^{M} [x_i(k)x_j(k) - \sum_{\nu=1}^{i} (k-m)^{i-\nu} x_\nu(M)x_j(k) - \sum_{\mu=1}^{j} (k-M)^{j-\nu} x_\mu(M)x_i(k)$$

$$+ \sum_{\nu=1}^{i} \sum_{\nu=1}^{j} x_\nu(M)x_\mu(M)(k-M)^{i+j-\nu-\mu}]$$

and

$$x_i(M) = \sum_{\nu=0}^{M-1} x_{i-1}(\nu).$$

Here it is assumed that $x_i(p)=0$ for $p>M$. This assumption is reasonable for practical problems.

When the order of the approximation k in (25) becomes equal to the order of the system the sets $[Z_1]$, $[Z_2]$, ... $[Z_n]$ becomes linearly dependent and hence

$$\det[\underline{H}] = 0 = \sum_{i=1}^{k} \sum_{j=1}^{k} (d*)^{i-1}(d)^{j-1} Q_{ij}. \tag{27}$$

Since

$$Q_{ij}^2 + \det[\underline{G}]_n Q_{ij}Q_{jj} = Q_{ii}Q_{jj} \tag{28}$$

and

$$\det[\underline{G}] = 0 \text{ when } \det[\underline{H}] = 0, \tag{29}$$

(27) can be rewritten as

$$\{ \sum_{i=1}^{n+1} \sqrt{Q_{ii}} (d*)^{n-i+1} \} \cdot \{ \sum_{j=1}^{n+1} \sqrt{Q_{jj}} (d)^{n-j+1} \} = 0 \tag{30}$$

From (30) the poles $z_i$ can be obtained from the solution of the following polynomial equation:

$$\sum_{i=1}^{n+1} \sqrt{Q_{ii}} (-1+z)^{n=j+1} = 0 \tag{31}$$

and the poles $s_i$ are obtained from $z_i$ through the transformation

$$s_i = \frac{1}{T} \ln [z_i]$$

Since the function to be approximated is considered to be real there will also be a complex conjugate set of poles $s_i^*$ as indicated by (30).

(6)

Once the poles $s_i$ or $z_i$ are determined the unknown constants $A_{ij}$ can be determined from a least squares procedure.

So far we have done the analysis for discrete systems. However the analysis for the continuous system will be done in section 6.

5. Error Analysis in the Presence of Noise.

When the signal $x(t)$ or the sequence $\{y(p)\}$ does not contain any noise then the poles $z_i$ given by (31) are the exact poles of the system. However if noise is present then they no longer represent the system poles. The objective of this present section is to find the error in the location of the poles due to additive noise.

Let $\{e(p)\}$ be the noisy sequence and let there exist an approximant of $\{e(p)\}$ and call it $\{y(p)\}$, which provides the best exponential approximation of the signal in the given subspace. The assumption made here is that the optimum approximant is unique.

For convenience let us call

$$e(p) = y(p) + \varepsilon r(p) \tag{32}$$

We now introduce the error sequence in the normalized form

$$[\{e(p) - y(p)\}] = \varepsilon\{r(p)\} \tag{33}$$

where $\varepsilon$ is a non-negative real number chosen to establish the equality

$$\sum_p \{r(p)\}^2 = \sum_p \{y(p)\}^2 \tag{34}$$

Clearly $\varepsilon$ represents the fractional error in the representation and is therefore a measure of the degree of approximation of $\{q(p)\}$. If $\{y(p)\}$ is the best approximant of $\{e(p)\}$, then $\{r(p)\}$ must be orthogonal to $\{y(p)\}$. Thus, in particular, $\varepsilon=0$ characterizes the perfect representation.

Now consider the following sequences:

(7)

$$[E_1] = \{e_1(p)\} = \{e(p)\}$$

$$[E_2] = \{e_2(p)\} = T\{e_1(p)\}$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$[E_{k+1}] = \{e_{k+1}(p)\} = T\{e_k(p)\} = T\{y_k(p) + \varepsilon r_k(p)\} \tag{35}$$

In (35) $\{y(p)\}$ and $\{r(p)\}$ are not known explicitly, nor is $\varepsilon$ for that matter. We begin with the computation of the $(n+1)X(n+1)$ gram matrix $[\underline{B}]$ for the sequence $[E_1]$, $[E_2]$, ..., $[E_n]$, i.e.

$$[\underline{B}] = [b_{ij} \triangleq <\{e_i(p)\}, \{e_j(p)\}>]$$

$$= \begin{Bmatrix} y_1(p) + \varepsilon r_1(p) \\ \cdot \\ \cdot \\ y_{n+1}(p) + \varepsilon r_{n+1}(p) \end{Bmatrix} \cdot \begin{Bmatrix} y_1(p) + \varepsilon r_1(p) \ldots \ldots y_{n+1}(p) + \varepsilon r_{n+1}(p) \end{Bmatrix} \tag{36}$$

Since $\{y(p)\}$ is a sequence of order n, the corresponding $[Y_1]$, ..., $[Y_{n+1}]$ sequences are linearly dependent. Thus there exists then a nonsingular matrix $[\underline{P}]$ such that

$$[\underline{Y}] \triangleq \begin{Bmatrix} \{y_1(p)\} \\ \cdot \\ \cdot \\ \cdot \\ \{y_{n+1}(p)\} \end{Bmatrix} = [\underline{P}] \begin{Bmatrix} \{\overline{y}_1\} \\ \cdot \\ \cdot \\ \{\overline{y}_n\} \\ 0 \end{Bmatrix} = [\underline{P}][\overline{\underline{Y}}] \tag{37}$$

From the theory of orthoganal transformation the sequences $\{\overline{y}_1(p)\}$, $\{\overline{y}_2(p)\}$, ..., $\{\overline{y}_n(p)\}$ must form an orthonormal set.

Similarly we obtain

$$[\underline{R}] = \begin{Bmatrix} \{r_1(p)\} \\ \cdot \\ \cdot \\ \cdot \\ \{r_{n+1}(p)\} \end{Bmatrix} = [\underline{P}] \begin{Bmatrix} \{\overline{r}_1(p)\} \\ \cdot \\ \cdot \\ \{\overline{r}_{n+1}(p)\} \end{Bmatrix} = [\underline{P}][\overline{\underline{R}}] \tag{38}$$

Observe in (38) the sequences $\{\overline{r}_1(p)\}$, ..., $\{\overline{r}_{n+1}(p)\}$ are not orthonormal to each other. Substitution of (37) and (38) into (36) yields

$$[\underline{B}] = [\underline{P}][\overline{\underline{Y}}+\epsilon\overline{\underline{R}}][\overline{\underline{Y}}+\epsilon\overline{\underline{R}}]^T[\underline{P}]^T$$

$$= [\underline{P}]\{[\overline{\underline{Y}}][\overline{\underline{Y}}]^T+\epsilon[\overline{\underline{R}}][\overline{\underline{Y}}]^T+\epsilon[\overline{\underline{Y}}][\overline{\underline{R}}]^T+\epsilon^2[\overline{\underline{R}}][\overline{\underline{R}}]^T\}[\underline{P}]^T \qquad (39)$$

Observe that

$$[\overline{\underline{Y}}][\overline{\underline{Y}}]^T = \begin{bmatrix} I_{n\times n} & \vdots & 0 \\ \hline 0 & \vdots & 0 \end{bmatrix}_{(n+1)\times(n+1)} \qquad (40)$$

where I is the nxn identity matrix and

$$[\overline{\underline{R}}][\overline{\underline{Y}}]^T = \left\{ \begin{array}{ccc} \{\overline{r}_1(p)\}\cdot\{\overline{y}_1(p)\} & \cdots & \{\overline{r}_1(p)\}\cdot\{\overline{y}_n(p)\} \\ \\ \{\overline{r}_{n+1}(p)\}\cdot\{\overline{y}_1(p)\} & \cdots & \{\overline{r}_{n+1}(p)\}\cdot\{\overline{y}_n(p)\} \end{array} \right\} \qquad (41)$$

By utilizing the theorem on determinant expansion as presented in Appendix A, we get

$$\det[\underline{B}] = (\det[\underline{P}])^2\cdot\det\left|\left|\begin{bmatrix} I_{n\times n} & \vdots & \epsilon\{\overline{r}_{n+1}(p)\}\cdot\{\overline{y}_1(p)\} \\ \hline 0 & \vdots & \epsilon^2\{\overline{r}_{n+1}(p)\}\cdot\{\overline{y}_{n+1}(p)\} \end{bmatrix}\right|\right|$$

$$+ \text{ terms of the order of } \epsilon^3 \text{ and higher}$$

$$= \epsilon^2\cdot||\{r_{n+1}(p)\}||^2\cdot(\det[\underline{P}])^2 + \Theta(\epsilon^3) + \cdots \qquad (42)$$

Thus it is clear that $\sqrt{\det[\underline{B}]}$ is of the same order of magnitude as the error measure $\epsilon$, i.e.

$$\det[\underline{B}] = \Theta(\epsilon) \qquad (43)$$

In particular, a neccessary and sufficient condition for perfect approximation of a signal of order n is that $\det[\underline{B}]$ must vanish.

In order to find the error in the location of the poles first the error in the diagonal cofactors of the gram matrix $[\underline{B}]$ has to be estimated. The diagonal cofactors are given by:

(9)

$$D_{ii} = \det \left| \left\{ \begin{matrix} \{y_1(p)+\varepsilon r_1(p)\} \\ \cdot \\ \cdot \\ \{y_{i-1}(p)+\varepsilon r_{i-1}(p)\} \\ \{y_{i+1}(p)+\varepsilon r_{i+1}(p)\} \\ \cdot \\ \cdot \\ \{y_{n+1}(p)+\varepsilon r_{n+1}(p)\} \end{matrix} \right\} \cdot \left\{ \begin{matrix} \{y_1(p)+\varepsilon r_1(p)\}\ldots\{y_{i-1}(p)+\varepsilon r_{i-1}(p)\} \\ \\ \{y_{i+1}(p)+\varepsilon r_{i+1}(p)\}\ldots\{y_{n+1}(p)+\varepsilon r_{n+1}(p)\} \end{matrix} \right\} \right| \quad (44)$$

Let $[\underline{S}_i]$ be the matrix which orthonormalizes the vectors, from

$\{y_1(p)\}\ldots\{y_{i-1}(p)\}\{y_i(p)\}\ldots\{y_{n+1}(p)\}$ to $\{\overline{y}_1(p)\}\ldots\{\overline{y}_{i-1}(p)\}\{\overline{y}_{i+1}(p)\}$

$\ldots\{\overline{y}_{n+1}(p)\}$. So we have

$$\left\{ \begin{matrix} \{y_1(p)\} \\ \cdot \\ \cdot \\ \{y_{i-1}(p)\} \\ \{y_{i+1}(p)\} \\ \cdot \\ \cdot \\ \{y_{n+1}(p)\} \end{matrix} \right\} = [\underline{S}_i] \cdot \left\{ \begin{matrix} \{\overline{y}_1(p)\} \\ \cdot \\ \cdot \\ \{\overline{y}_{i-1}(p)\} \\ \{\overline{y}_{i+1}(p)\} \\ \cdot \\ \cdot \\ \{\overline{y}_{n+1}(p)\} \end{matrix} \right\} \triangleq [\underline{S}_i][\overline{\underline{Y}}_i] \quad (45)$$

$$\left\{ \begin{matrix} \{r_1(p)\} \\ \cdot \\ \cdot \\ \{r_{i-1}(p)\} \\ \{r_{i+1}(p)\} \\ \cdot \\ \cdot \\ \{r_{n+1}(p)\} \end{matrix} \right\} = [\underline{S}_i] \cdot \left\{ \begin{matrix} \{\overline{r}_1(p)\} \\ \cdot \\ \cdot \\ \{\overline{r}_{i-1}(p)\} \\ \{\overline{r}_{i+1}(p)\} \\ \cdot \\ \cdot \\ \{\overline{r}_{n+1}(p)\} \end{matrix} \right\} \triangleq [\underline{S}_i][\overline{\underline{R}}_i] \quad (46)$$

Observe that the vectors $\{\overline{r}_1(p)\} \ldots \{\overline{r}_{n+1}(p)\}$ are not a set of

orthonormal vectors. Substitution of (45) and (46) into (44) yields

$$D_{ii} = \det \left[ [\underline{S}_i][\overline{\underline{Y}}_i+\varepsilon\overline{\underline{R}}_i][\overline{\underline{Y}}_i+\varepsilon\overline{\underline{R}}_i]^T[\underline{S}_i]^T \right]$$

$$= (\det[\underline{S}_i])^2 \cdot \det \left[ [\overline{\underline{Y}}_i][\overline{\underline{Y}}_i]^T+\varepsilon[\overline{\underline{R}}_i][\overline{\underline{Y}}_i]^T+\varepsilon[\overline{\underline{Y}}_i][\overline{\underline{R}}_i]^T+\varepsilon^2[\overline{\underline{R}}_i][\overline{\underline{R}}_i]^T \right] \quad (47)$$

(10)

By utilizing

$$[\overline{\underline{Y}}_i][\overline{\underline{Y}}_i]^T = [I]_{n \times n}$$

and applying the theorem on determinant expansion [as presented in Appendix A] we obtain

$$D_{ii} = Q_{ii} + 2\epsilon(\det[\underline{S}_i])^2 \cdot \left\{ \sum_{\substack{j=1 \\ i \neq j}}^{n+1} \sum_{p=0}^{\infty} \{\overline{y}_j(p)\} \cdot \{r_j(p)\} \right\} \tag{48}$$

where $Q_{ii}$ are the diagonal cofactors of the gram matrix formed by the sets $[Y_1]$, $[Y_2]$ ... $[Y_{n+1}]$ and is given by (26). The underlying assumption in (48) is of course we have the right order of the system n, i.e.

$$\det[\underline{G}] = 0. \tag{49}$$

Observe that (48) implies continuous differentiability of the diagonal cofactors $D_{ii}$ with respect to the error $\epsilon$, so that

$$\lim_{\epsilon \to 0} D_{ii} = Q_{ii} \quad \text{for all i.} \tag{50}$$

Let $z_i'$ be the approximate poles that are obtained by utilizing the diagonal cofactors $D_{ii}$ instead of $Q_{ii}$ in (31). So the poles $z_i'$ are the roots of the following equation

$$\sum_{i=1}^{n+1} \sqrt{D_{ii}}(-1+z_i')^{n+1-i} = 0 \tag{51}$$

Substitution of (48) into (51) illustrates that the approximate poles $z_i'$ are related to the exact poles $z_i$ by the relationship

$$z_i' = z_i + \epsilon F_i + \Theta(\epsilon^2) \tag{52}$$

where $F_i$ are certain constants. The approximate poles $z_i'$ are again a continuous function of $\epsilon$. So the error in the approximation $(z_i' - z_i)$ is a continuous function of $\epsilon$ and approaches zero uniformly as $\epsilon \to 0$.

(11)

This is the unique feature of this technique. The continuous dependence of the error in the pole locations on $\varepsilon$ is not guaranteed by other optimization techniques and the classical nonlinear methods based on Wiener filter theory [4-5].

6. <u>Continuous Version of the Pencil of Function Method.</u>

In the previous sections the approximation problem involved discretizing a continuous function which was then processed digitally. A strictly continuous version is also available. The problem is to approximate the continuous function $x(t)$ in the form given by (1), i.e.

$$x(t) = \sum_{i=1}^{\ell} \sum_{k=1}^{m_i} A_{ik} \{t\}^{k-1} \exp\{ s_i t\}$$

In this case we want to estimate $s_i$ instead of $\exp\{s_i T\}$. For the continuous case the T operator of the previous sections is replaced by the reverse time integral operator so that

$$x_{i+1}(t) = \int_t^{\infty} x_i(f) df \tag{53}$$

and $x_1(t) = x(t)$

The two sided Laplace Transform of (53) is given by

$$X_{i+1}(s) = \int_{-\infty}^{\infty} X_{i+1}(t) e^{-st} dt = \frac{X_1(s)}{s^i} \tag{54}$$

and

$$X_1(s) = \sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \frac{A_{ij}}{s^i} \frac{(j-1)!}{(s-s_i)^j} = \frac{N(s)}{\prod_{i=1}^{\ell} (s-s_i)^{m_i}} \tag{55}$$

Next we consider the pencil of functions defined by

$$\{x_1(t) - \lambda x_2(t)\}, \ \{x_2(t) - \lambda x_3(t)\}, \ \ldots, \ \{x_n(t) - \lambda x_{n+1}(t)\} \tag{56}$$

Since for this problem it is simpler to deal with the Laplace transform, attention is focused on the set of pencil of functions defined by

$$\{X_1(s) - \lambda X_2(s)\}, \ \{X_2(s) - \lambda X_3(s)\}, \ldots, \ \{X_n(s) - \lambda X_{n+1}(s)\} \tag{57}$$

(12)

The above set of pencil of functions displays some interesting properties if checked for linear independence. Checking for linear independence, we write

$$\sum_{k=1}^{n} [a_k \{X_k(s) - \lambda X_{k+1}(s)\}] = 0 \tag{58}$$

Substitution of (54) into (58) yields

$$\sum_{k=1}^{n} s^{n-k-1} [a_k(s-\lambda)X(s)] = 0$$

or

$$\sum_{k=1}^{n} s^{n-k} [a_k(s-\lambda) \frac{N(s)}{\prod_{i=1}^{\ell}(s-s_i)^{m_i}}] = 0 \tag{59}$$

Equation (59) represents a polynomial in s of the order n. Since $1, s, s^2, \ldots, s^{n-1}$ span a n dimensional subspace, this set is linearly independent and hence the set of n coefficients $a_k$ must be identically zero if (59) is to be satisfied for all values of s. The above holds as long as $\lambda \neq \lambda_i$. For $\lambda = \lambda_i$, it turns out that (59) has a root $s = \lambda_i$ which can be factored out. The set $1, s, s^2, \ldots, s^{n-2}$ spans a (n-1) dimensional subspace. Nevertheless there remains n set of coefficients $a_k$. It is now possible for (59) to be satisfied with at least one non-zero coefficient. This implies linear dependence of the set (57) when $\lambda = \lambda_i$. So if $\lambda$ is a system pole then the gram determinant of the set of functions in (57) will be zero. The gram determinant of the given set in (56) is defined by

$$[\underline{F}] = \begin{bmatrix} <x_1-\lambda x_2, x_1-\lambda x_2> & \cdots & \cdots & <x_1-\lambda x_2, x_n-\lambda x_{n+1}> \\ \vdots & & & \vdots \\ \vdots & \cdots \cdots \cdots \cdots \cdots & \vdots \\ \vdots & & & \vdots \\ <x_n-\lambda x_{n+1}, x_1-\lambda x_2> & \cdots \cdots & <x_n-\lambda x_{n+1}, x_n-\lambda x_{n+1}> \end{bmatrix} \tag{60}$$

(13)

where

$$\langle x_i, x_j \rangle = \int_0^\infty x_i(t)x_j(t)dt \tag{61}$$

Thus $\det[\underline{F}] = 0$ if $\lambda$ is a system pole. After some tedious algebraic manipulations it can be shown that

$$\det[\underline{F}] = \left\{ \sum_{i=1}^{n+1} \sqrt{\Delta_{ii}} \lambda^{n-i+1} \right\}\left\{ \sum_{j=1}^{n+1} \sqrt{\Delta_{jj}} \lambda^{n-j+1} \right\} = 0 \tag{62}$$

where $\Delta_{kk}$ is the kth diagonal cofactor in the grammian $[\underline{F}^1]$ which is defined by

$$[\underline{F}^1] = \begin{bmatrix} \langle x_1, x_1 \rangle & \cdots\cdots\cdots & \langle x_1, x_{n+1} \rangle \\ \vdots & & \vdots \\ \vdots & \cdots\cdots\cdots & \vdots \\ \vdots & & \vdots \\ \langle x_{n+1}, x_1 \rangle & & \langle x_{n+1}, x_{n+1} \rangle \end{bmatrix} \tag{63}$$

The polynomial equation whose roots are the pole locations, is then given by

$$\sum_{i=1}^{n+1} \sqrt{\Delta_{ii}} \lambda^{n-i+1} = 0 \tag{64}$$

After the poles $s_i$ have been obtained the residues are obtained by a least squares fashion.

## 7. Examples

As an example consider the transient response of a conducting pipe tested at the ATHAMAS-I Electromagnetic Pulse (EMP) simulator. The conducting pipe is 10m long and 1m in diameter. Hence the true resonance of the pipe is expected to be in the neighborhood of 14MHz. Also the pipe has been excited in such a way that it is reasonable to expect only odd harmonics at the scattered fields. The data which have been measured are the integral of the electric field and hence is available in terms of a voltage. Thus, in addition to the frequencies of the conducting pipe one should also observe a very dominant low frequency pole. The same transient data as depicted in Figure 1 is used for

(14)

analysis. The results for a fifth and a seventh order system are as
follows:

For n = 5, the poles in radians/sec are

$$(-0.0029 \pm j0.083) \times 10^9 \qquad (=13.33\text{MHz})$$

$$(-0.0428 \pm j0.217) \times 10^9 \qquad (=35.20 \text{ MHz})$$

$$(-0.0098 \qquad ) \times 10^9 \qquad (= 1.56 \text{ MHz})$$

For n = 7, the poles in radians/sec are

$$(-0.0058 \pm j0.084) \times 10^9 \qquad (=13.40 \text{ MHz})$$

$$(-0.0270 \pm j0.219) \times 10^9 \qquad (=35.10 \text{ MHz})$$

$$(-0.0270 \pm j0.550) \times 10^9 \qquad (=87.60 \text{ MHz})$$

$$(-0.0012 \qquad ) \times 10^9 \qquad (= 0.19 \text{ MHz})$$

It is interesting to observe that the real pole due to the
integrator has been obtained. This pole is a very dominant pole as
the data were recorded after having passed through an integrator.
The above results display a dynamic range of approximately 1000:1
for the values of poles of the conducting pipe.

Next the data were differentiated to get rid of the undesirable
dominang pole of the integrator. The differentiation was done
numerically. For a fourth and sixth order system the above results
have been recalculated as follows:

For n = 4, the poles in radians/sec are

$$(-0.0026 \pm j0.086) \times 10^9 \qquad (=13.70 \text{ MHz})$$

$$(-0.0480 \pm j0.235) \times 10^9 \qquad (=37.47 \text{ MHz})$$

For n = 6, the poles in radians/sec are

$$(-0.005 \pm j0.083) \times 10^9 \qquad (=13.23 \text{ MHz})$$

$$(-0.034 \pm j0.221) \times 10^9 \qquad (=35.59 \text{ MHz})$$

$$(-0.071 \pm j0.406) \times 10^9 \qquad (= 65.9 \text{ MHz})$$

(15)

Here a good approximation to the poles has been obtained with only

four poles. Also, there seems to be a good agreement in the pole

locations obtained from the original integrated data and the numerically

differentiated data. It is also interesting to observe that indeed

the poles are occurring approximately at odd harmonics of the fundamental.

In figure 2, the true numerically differentiated data are plotted

against the reconstructed response of a sixth order system. The

plot has been normalized to unity amplitude. It is interesting to

note that there is a close agreement even in the very early times of

the two waveforms.

## 8. Conclusion

The pencil of function method has been applied to the approximation

of an arbitrary function by a sum of complex exponentials. The

important features of this technique are the natural insensitivity

to noise and the continuous dependence of the pole locations to the

integrated square error. Finally examples have been presented to

illustrate the stability of the pole locations yielded by the pencil

of function method.

## 9. References

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## 10. Appendix

Let $\underline{A}$ and $\underline{B}$ be two square N×N matrices. Let the columns of $\underline{A}$ and $\underline{B}$ be represented by $A_i$ and $B_i$ for $i=1,2,\ldots,N$. Denote

$$C^1_{\nu_i} = A_i \quad \text{if } \nu_i=1$$

$$\qquad = B_i \quad \text{if } \nu_i=0$$

and the matrix constituted by the columns of

$$C^1_{\nu_1}, \ C^2_{\nu_2}, \ C^3_{\nu_3}, \ \ldots, \ C^N_{\nu_N} \quad \text{as} \quad \underline{C}_{\nu_1,\nu_2,\ldots,\nu_N}.$$

Then

$$\det\left|[\underline{A}+\underline{B}]\right| = \sum_{m=0}^{N} \det\left|\underline{C}_{\nu_1,\nu_2,\ldots,\nu_N}\right|$$

where in the second summation exactly m of the $\nu_i$'s equal 1 and the rest equal 0.

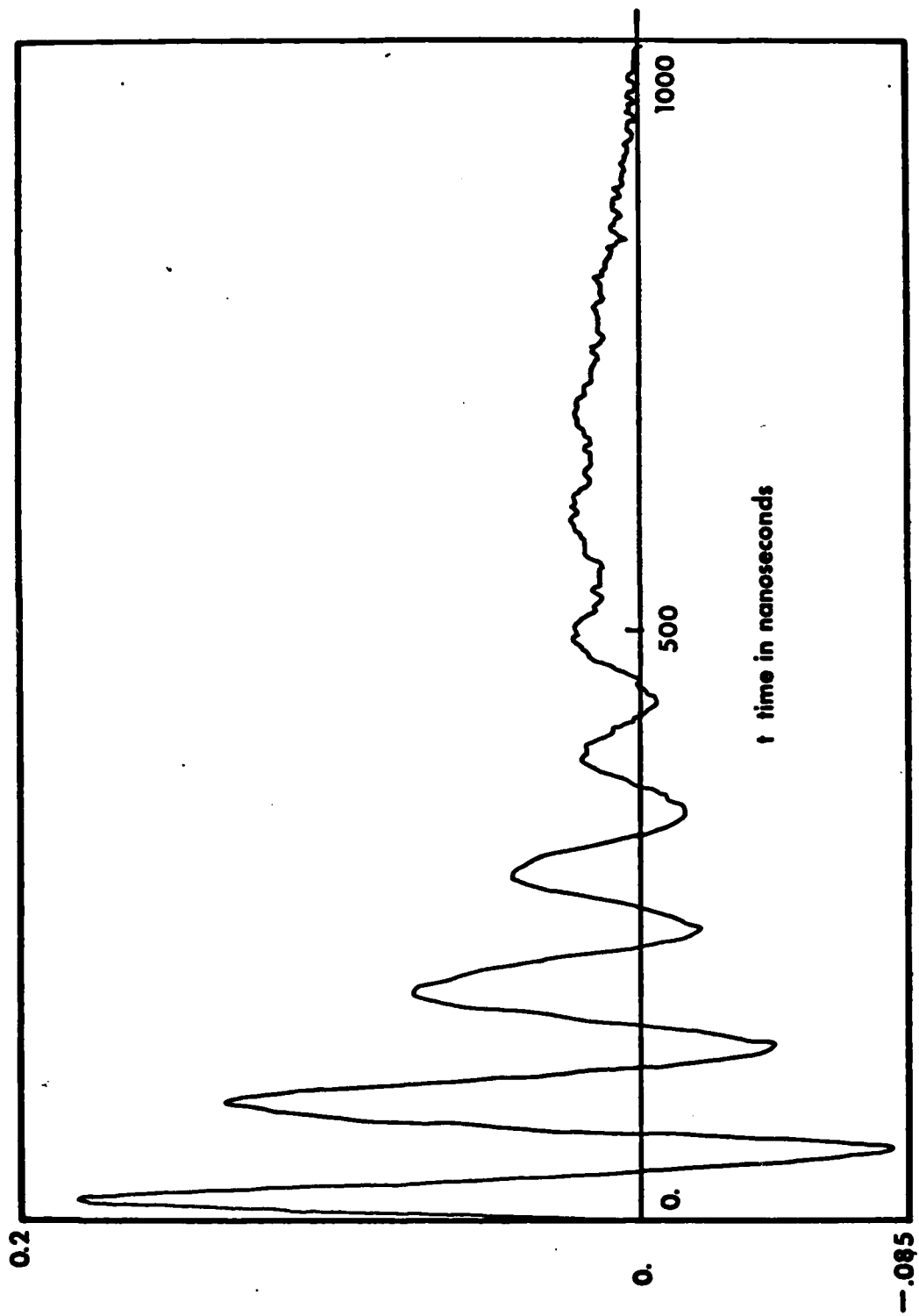This is a standard result of determinant expansion.

Figure 1. Transient Response of a Conducting Pipe, Measured at the ATHAMAS-I EMP Simulator.
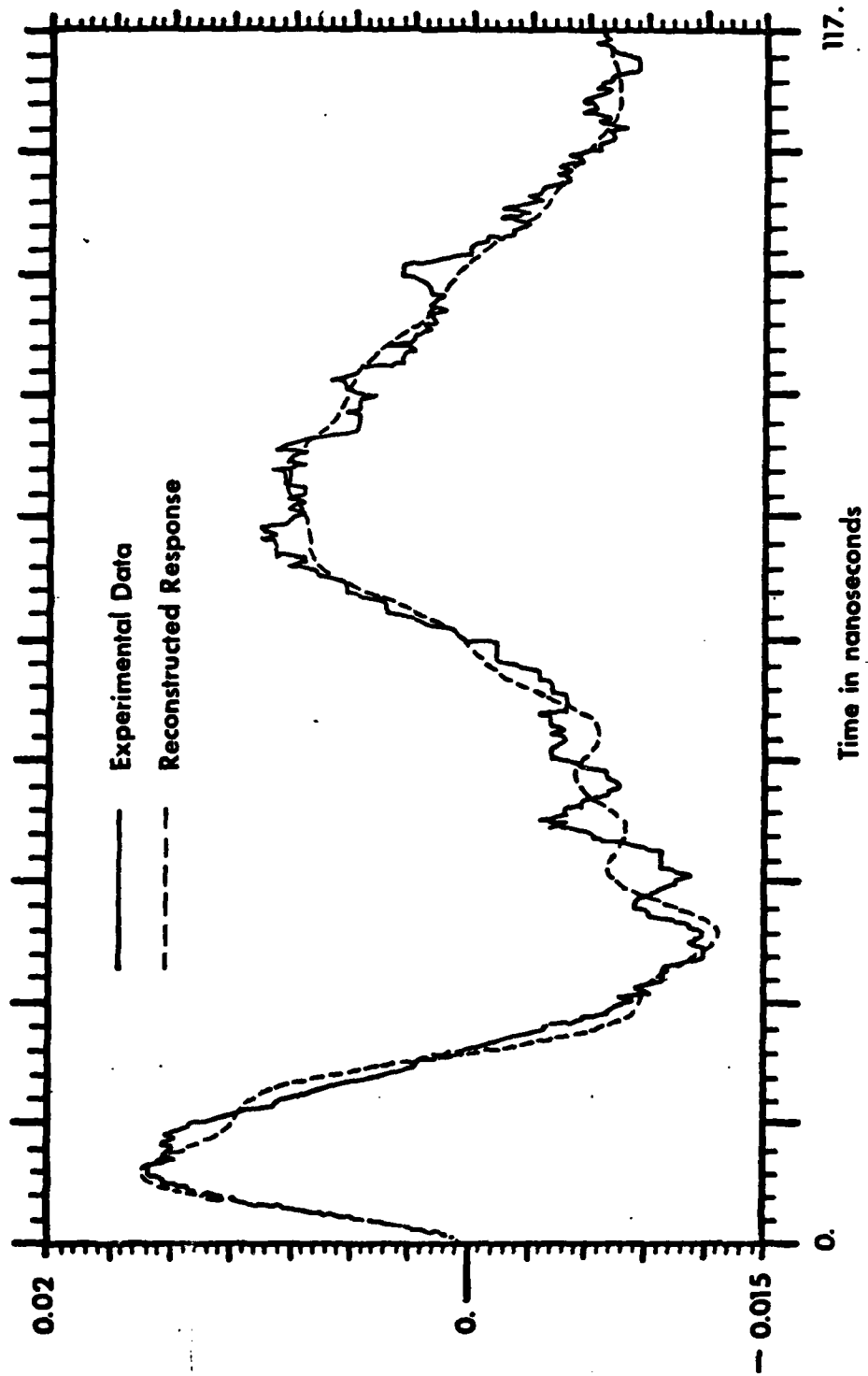
Figure 2. True Response Vs. Reconstructed Response of a Sixth Order System for a 10m long 1m Diameter Conducting Pipe.

# DAT FILM